

On the inheriting of the property C_π by some normal subgroups *

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Abstract

In the paper we prove that the Hall property C_π is inherited by normal subgroups which index is a π' -number.

Introduction

Let π be a set of primes. We denote by π' the set of all primes not in π , by $\pi(n)$ the set of prime divisors of a positive integer n , while for a finite group G by $\pi(G)$ we denote $\pi(|G|)$. A positive integer n with $\pi(n) \subseteq \pi$ is called a π -number, while a group G with $\pi(G) \subseteq \pi$ is called a π -group. A subgroup H of G is called a π -Hall subgroup, if $\pi(H) \subseteq \pi$ by $\pi(|G : H|) \subseteq \pi'$. According to [1] we shall say that G has the property E_π (or briefly $G \in E_\pi$), if G contains a π -Hall subgroup. If we also have that every two π -Hall subgroups are conjugate, then we shall say that G has the property C_π ($G \in C_\pi$). If we have further that each π -subgroup of G is contained in a π -Hall subgroup, then we shall say that G has the property D_π ($G \in D_\pi$). A group with the property E_π (C_π , D_π) we shall call also an E_π - (respectively a C_π -, a D_π -) group. The expression (mod CFSG) means that the corresponding result is proved by using the classification of finite simple groups.

Assume that a set π is fixed. It is proved that the class of all D_π -groups is closed under homomorphic images, normal subgroups (mod CFSG, [5, Theorem 7.7]) and extensions (mod CFSG, [5, Theorem 7.7]). The class of E_π -groups is also known to be closed under normal subgroups and homomorphic images (see Lemma 6(1)), but, in general, it is not closed under extensions (see Example 1). The class of C_π -groups is closed under homomorphic images (mod CFSG, see Lemma 9) and extensions (see Lemma 7), but, in general, is not closed under normal subgroups (see Example 2). Nevertheless, while proving statements about Hall properties one need to know in which cases an extension of an E_π -group by an E_π -group has the property E_π , and a normal subgroup of a C_π -group has the property C_π . In the present paper we prove by using the classification of finite simple groups

Theorem 1. (mod CFSG) *Let π be a set of primes, H be a π -Hall, and A be a normal subgroups of a C_π -group G . Then $HA \in C_\pi$.*

Since every π' -group has the property C_π and the class of C_π -groups is closed under extensions, the following statement is immediate from Theorem 1 (cf. Example 2).

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Corollary 2. *Let π be a set of primes and A be a normal subgroup of G , which index in G is a π' -number. Then $G \in C_\pi$ if and only if $A \in C_\pi$.*

The proof of Theorem 1 is based on the following result about the number of classes of π -Hall subgroups in finite simple groups.

Theorem 3. (mod CFSG) *Let π be a set of primes and S a finite simple group. Then there exist at most 4 classes of conjugate π -Hall subgroups of S . More precisely, the following statements hold.*

- (1) *If $2 \notin \pi$, then S has at most one class of conjugate π -Hall subgroups.*
- (2) *If $2 \in \pi$, and $3 \notin \pi$, then S has at most two classes of conjugate π -Hall subgroups.*
- (3) *If $2, 3 \in \pi$, then S has at most four classes of conjugate π -Hall subgroups.*

Corollary 4. *Let π be a set of primes and S a finite simple E_π -group. Then if a positive integer k is not greater than the number of classes of conjugate π -Hall subgroups of S , then k is a π -number.*

In view of Theorem 1 note that the authors do not know any counter example to the following conjecture.

Conjecture 5. *Let π be a set of primes, G be a finite C_π -group, and A be its (not necessary normal) subgroup, containing a π -Hall subgroup of G . Then $A \in C_\pi$.*

In the conjecture the condition that A contains a π -Hall subgroup of G cannot be weakened by substituting it with condition that the index of A is a π' -number. Indeed, consider $B_3(q) \simeq \text{P}\Omega_7(q)$, where $q - 1$ is divisible by 12 and is not divisible by 24. In view of [5, Lemma 6.2] the group $\text{P}\Omega_7(q)$ is a $C_{\{2,3\}}$ -group and its $\{2, 3\}$ -Hall subgroup is contained in a monomial subgroup. On the other hand the group $\Omega_7(2)$ is known to be embeddable into $\text{P}\Omega_7(q)$ and, under above mentioned conditions on q , its index is not divisible by 2 and by 3. Although $\Omega_7(2)$ does not contain $\{2, 3\}$ -Hall subgroups, i. e., it is not even a $E_{\{2,3\}}$ -group.

1 Notations and preliminary results

By π we always denote a set of primes, and the term group always means a finite group.

For a group G , a G -class of π -Hall subgroups is a class of conjugate π -Hall subgroups of G . Let A be a subnormal subgroup of a E_π -group G . A subgroup of A of type $H \cap A$, where H is a π -Hall subgroup of G , we shall call a G -induced π -Hall subgroup of A . Thus a set $\{(H \cap A)^x \mid x \in A\}$, where H is a π -Hall subgroup of G is called an A -class of G -induced π -Hall subgroups. By $k_\pi^G(A)$ we denote the number of all A -classes of G -induced π -Hall subgroups. Assume also that $k_\pi(G) = k_\pi^G(G)$ is the number of conjugacy classes of π -Hall subgroups of G . It is clear that $k_\pi^G(A) \leq k_\pi(G)$.

Let $A = A_1 \times \cdots \times A_s$ and for each $i = 1, \dots, s$ by \mathcal{K}_i we denote an A_i -class of π -Hall subgroups of A_i . The set

$$\mathcal{K}_1 \times \cdots \times \mathcal{K}_s = \{\langle H_1, \dots, H_s \rangle \simeq H_1 \times \cdots \times H_s \mid H_i \in \mathcal{K}_i, i = 1, \dots, s\}$$

is called a *product of classes* $\mathcal{K}_1, \dots, \mathcal{K}_s$. Clearly $\mathcal{K}_1 \times \cdots \times \mathcal{K}_s$ is an A -class of π -Hall subgroups of A . It is also clear that if A is a normal subgroup of G , then each A -class of G -induced π -Hall subgroups is a product of some A_1 -, \dots , A_s -classes of G -induced π -Hall subgroups. The reverse statement, in general, is not true.

The following statements are known and their proof does not use the classification of finite simple groups.

Lemma 6. *Let A be a normal subgroup of G . Then the following statements hold.*

(1) *If H is a π -Hall subgroup of G , then $H \cap A$ is a π -Hall subgroup of A , and HA/A is a π -Hall subgroup of G/A .*

(2) *If all factors of a subnormal series of G are either π - or π' - groups, then $G \in D_\pi$.*

Lemma 7. (S.A.Chunihin, see also [1, Theorems C1 and C2]) *Let A be a normal subgroup of G . If A and G/A has the property C_π , then $G \in C_\pi$.*

Example 1. Let $\pi = \{2, 3\}$. Let $G = \text{GL}_3(2) = \text{SL}_3(2)$ be a group of order $168 = 2^3 \cdot 3 \cdot 7$. From [7, Theorem 1.2] or [2] it follows that G has exactly two classes of π -Hall subgroups with representatives

$$\left(\begin{array}{cc} \boxed{\text{GL}_2(2)} & * \\ 0 & \boxed{1} \end{array} \right) \text{ and } \left(\begin{array}{cc} \boxed{1} & * \\ 0 & \boxed{\text{GL}_2(2)} \end{array} \right).$$

The first one consists of line stabilizers in the natural representation of G , and the second one consists of plain stabilizers. The map $\iota : x \in G \mapsto (x^t)^{-1}$, where x^t means the transposed matrix to x , is an automorphism of order 2 of G . It interchange classes of π -Hall subgroups. If $\widehat{G} = G \rtimes \langle \iota \rangle$ is a natural semidirect product, then $N_{\widehat{G}}(H) = N_G(H) = H$ for each π -Hall subgroup H of G , since each element from $\widehat{G} \setminus G$ interchanges classes of π -Hall subgroups, like ι does, and H is maximal. The group \widehat{G} has order $2^4 \cdot 3 \cdot 7$ and has no the property E_π . Indeed, if there would exist a π -Hall subgroup \widehat{H} , then by Lemma 6(1) there would also exist a π -Hall subgroup H of G , with $H = \widehat{H} \cap G$. But this implies $\widehat{H} \leq N_{\widehat{G}}(H) = H$, and so $|\widehat{H}| \leq 2^3 \cdot 3$, that contradicts the condition that \widehat{H} is a π -Hall subgroup.

Example 2. Let $\pi = \{2, 3\}$. Let $G = \text{GL}_5(2) = \text{SL}_5(2)$ be a group of order $99999360 = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$. Let $\iota : x \in G \mapsto (x^t)^{-1}$ and $\widehat{G} = G \rtimes \langle \iota \rangle$ be a natural semidirect product. From [7, Theorem 1.2] it follows that there exist π -Hall subgroups of G , and every such a subgroup is a stabilizer of a series of subspaces $V = V_0 < V_1 < V_2 < V_3 = V$, where V is a natural module of G , and $\dim V_k/V_{k-1} \in \{1, 2\}$ for every $k = 1, 2, 3$. Therefore, there are three conjugacy classes of π -Hall subgroups of G with representatives

$$H_1 = \left(\begin{array}{cc} \boxed{\text{GL}_2(2)} & * \\ & \boxed{1} \\ 0 & \boxed{\text{GL}_2(2)} \end{array} \right),$$

$$H_2 = \left(\begin{array}{cc} \boxed{1} & * \\ \boxed{\text{GL}_2(2)} & \\ 0 & \boxed{\text{GL}_2(2)} \end{array} \right), \text{ and } H_3 = \left(\begin{array}{cc} \boxed{\text{GL}_2(2)} & * \\ & \boxed{\text{GL}_2(2)} \\ 0 & \boxed{1} \end{array} \right).$$

Note that $N_G(H_k) = H_k$, $k = 1, 2, 3$, since H_k is parabolic. The class containing H_1 is ι -invariant. So Frattini argument implies that $\widehat{G} = GN_{\widehat{G}}(H_1)$, whence $|N_{\widehat{G}}(H_1) : N_G(H_1)| = 2$ and so $N_{\widehat{G}}(H_1)$ is a π -Hall subgroup of \widehat{G} . ι interchanges classes containing H_2 and H_3 . So, as in the previous example, these subgroups are not contained in π -Hall subgroups of \widehat{G} . Thus \widehat{G} contains exactly one class of π -Hall subgroups, therefore has the property C_π , while its normal subgroup G has no this property.

Lemma 8. ([3, Lemma 3], mod CFSG) *Let A be a normal subgroup of an E_π -group G , B be a subgroup of G such that $A \leq B$ and $B/A \in E_\pi$. Then $B \in E_\pi$.*

It is immediate from Lemma 8 that the property C_π is preserved under homomorphisms and we shall give the proof here for completeness. Note that since Lemma 8 is proved by using the classification of finite simple groups, then Lemma 9 is also proved by using the classification of finite simple groups.

Lemma 9. (mod CFSG) *Let A be a normal subgroup of a C_π -group G . Then $G/A \in C_\pi$.*

Proof. Let $G/A = \overline{G}$. Since all π -Hall subgroups of G are conjugate, it is enough to prove that for every π -Hall subgroup \overline{K} of \overline{G} there exists a π -Hall subgroup U of G such that $UA/A = \overline{K}$. The existence of such a subgroup U follows from Lemma 8. Indeed, for K being the complete preimage of \overline{K} , the group $K/A = \overline{K}$ has the property E_π . So K has the property E_π , and its π -Hall subgroup U is a π -Hall subgroup of G . By construction, $UA/A = \overline{K}$. \square

Lemma 10. *Let A be a normal and H be a π -Hall subgroups of a C_π -group G . Then each of groups $N_G(HA)$ and $N_G(H \cap A)$ has the property C_π .*

Proof. Let K be a π -Hall subgroup of $N_G(HA)$. Since $HA \trianglelefteq N_G(HA)$ and $|N_G(HA) : HA|$ is a π' -number, then $K \leq HA$ and $KA = HA$. If $x \in G$ is such that $K = H^x$, then $(HA)^x = H^x A = KA = HA$ and thus $x \in N_G(HA)$. Therefore $N_G(HA) \in C_\pi$.

Assume now that K is a π -Hall subgroup of $N_G(H \cap A)$. Then $K(H \cap A) = K$ and $K \cap A = H \cap A$. If $x \in G$ is such that $K = H^x$, then $(H \cap A)^x = H^x \cap A = K \cap A = H \cap A$ and thus $x \in N_G(H \cap A)$. Therefore $N_G(H \cap A) \in C_\pi$. \square

Lemma 11. *Let A be a normal, H be a π -Hall subgroup of a group G and $HAC_G(A) \trianglelefteq G$ (this condition is satisfied, if $HA \trianglelefteq G$). Then an A -class of π -Hall subgroups is a class of G -induced π -Hall subgroups if and only if it is H -invariant.*

Proof. If K is a π -Hall subgroup of G , then $K \leq HAC_G(A)$ holds and so $KAC_G(A) = HAC_G(A)$. Since the A -class $\{(K \cap A)^x \mid x \in A\}$ is K -invariant, it is invariant under $HAC_G(A) = KAC_G(A)$ and hence under H .

Now we prove the inverse statement. Without lost of generality we may assume that $G = HA$. Let U be a π -Hall subgroup of A such that $\{U^x \mid x \in A\}$ is H -invariant. Then Frattini argument implies $H \leq N_G(U)A$ and so $G = N_G(U)A$. Note that

$$N_G(U)/N_A(U) = N_G(U)/N_G(U) \cap A \simeq N_G(U)A/A = G/A = HA/A$$

is a π -group. Thus $N_G(U)$ has a normal series

$$N_G(U) \geq N_A(U) \geq U \geq 1,$$

each factor of this series is either a π - or a π' - group. By Lemma 6(2) $N_G(U)$ has the property D_π . Let K be its π -Hall subgroup. Clearly $U \leq K$. More over K is a π -Hall subgroup of G , since

$$|K| = |N_G(U)|_\pi = |N_G(U)/N_A(U)|_\pi |N_A(U)|_\pi = |HA/A|_\pi |U| = |HA/A| |H \cap A| = |H|.$$

Therefore, $U = K \cap A$ is a G -induced π -Hall subgroup. \square

Lemma 12. *Let A be a normal, H be a π -Hall subgroups of a C_π -group G and $HA \trianglelefteq G$. Then $k_\pi^G(A) = k_\pi^{HA}(A)$.*

Proof. Since HA is a normal subgroup of G , each π -Hall subgroup of G is contained in HA . By Lemma 10 $HA \in C_\pi$, and so the equality $k_\pi^G(A) = k_\pi^{HA}(A)$ holds. \square

Lemma 13. *Let A be a normal, H be a π -Hall subgroup of a E_π -group G and $HA \trianglelefteq G$. Then the following statements are equivalent.*

- (1) $k_\pi^G(A) = 1$.
- (2) $HA \in C_\pi$.
- (3) Every two π -Hall subgroups of G are conjugate by an element of A .

Proof. (1) \Rightarrow (2). If K is a π -Hall subgroup of HA , then by (1) groups $H \cap A$ and $K \cap A$ are conjugate in A . We may assume that $H \cap A = K \cap A$. Then H and K are contained in $N_{HA}(H \cap A)$. By Frattini argument we have $HA = N_{HA}(H \cap A)A$. So

$$N_{HA}(H \cap A)/N_A(H \cap A) = N_{HA}(H \cap A)/N_{HA}(H \cap A) \cap A \simeq N_{HA}(H \cap A)A/A = HA/A$$

is a π -group. Thus $N_{HA}(H \cap A)$ has a normal series

$$N_{HA}(H \cap A) \geq N_A(H \cap A) \geq H \cap A \geq 1,$$

each factor of this series is either a π - or a π' - group, and, by Lemma 6(2) has the property D_π . In particular, H and K are conjugate in $N_{HA}(H \cap A)$.

(2) \Rightarrow (3) and (3) \Rightarrow (1) are evident. \square

Lemma 14. *Let $A = A_1 \times \dots \times A_s$ be a normal and H be a π -Hall subgroups of G . Assume also that subgroups A_1, \dots, A_s are normal in G and $G = HAC_G(A)$. Then $k_\pi^G(A) = k_\pi^G(A_1) \dots k_\pi^G(A_s)$.*

Proof. Two π -Hall subgroups P and Q of A are conjugate in A if and only if π -Hall subgroups $P \cap A_i$ and $Q \cap A_i$ of A_i are conjugate in A_i for each $i = 1, \dots, s$. To prove the lemma it is enough to show that the product of every A_1 -, \dots , A_s - classes of G -induced π -Hall subgroups is an A -class of again G -induced π -Hall subgroups. Let U_1, \dots, U_s be G -induced π -Hall subgroups of A_1, \dots, A_s respectively. We shall show that

$$U = \langle U_1, \dots, U_s \rangle = U_1 \times \dots \times U_s$$

is a G -induced π -Hall subgroup of A . By Lemma 11 it is enough to show that for each $h \in H$ there exists $a \in A$ with $U^h = U^a$. Since $U_i = K_i \cap A_i$ for a suitable π -Hall subgroup K_i of G , then $\{U_i^{x_i} \mid x_i \in A_i\}$ is invariant under K_i and so under $K_i A = HA$. In particular, $U_i^h = U_i^{a_i}$ for some $a_i \in A_i$. Thus

$$U^h = U_1^h \times \dots \times U_s^h = U_1^{a_1} \times \dots \times U_s^{a_s} = U_1^a \times \dots \times U_s^a = U^a,$$

where $a = a_1 \dots a_s \in A$. \square

Lemma 15. *Let $A = A_1 \times \dots \times A_s$ be a normal, H be a π -Hall subgroups of G . Assume that G acts transitively by conjugation on the set $\{A_1, \dots, A_s\}$ and $G = HAC_G(A)$. Then $k_\pi^G(A) = k_\pi^G(A_1) = \dots = k_\pi^G(A_s)$.*

Proof. We shall show that if $x, y \in G$ are in the same coset of G by $N_G(A_1)$, then for each G -induced π -Hall subgroup U_1 of A_1 subgroups U_1^x and U_1^y are conjugate in $A_i = A_1^x = A_1^y$. It is enough to show that subgroups U_1 and U_1^t , where $t = xy^{-1} \in N_G(A_1)$, are conjugate in A_1 . Let $t = ach$, $a \in A$, $c \in C_G(A)$, $h \in H$. Since subgroups U_1 and U_1^{ac} are conjugate

in A_1 , we need to show that U_1 and U_1^h are conjugate in A_1 . Since $ac \in N_G(A_1)$, then h normalizes also the subgroup A_1 . Let $U_1 = U \cap A_1$ for some G -induced π -Hall subgroup U of A . Let \mathcal{K} be an A -class, containing U , and let $\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_s$, where \mathcal{K}_i is an A_i -class of G -induced π -Hall subgroups. Clearly $U_1 \in \mathcal{K}_1$. Since, according to Lemma 11, the class \mathcal{K} is H -invariant, then H acts on the set $\{\mathcal{K}_1, \dots, \mathcal{K}_s\}$. An element h normalizes A_1 , so stabilizes the A_1 -class \mathcal{K}_1 . In particular, subgroups U_1 and U_1^h are in \mathcal{K}_1 and so are conjugate in A_1 .

Let $f \in H$ and $A_1^f = A_i$. For an A_1 -class of π -Hall subgroups \mathcal{K}_1 define an A_i -class \mathcal{K}_1^f , assuming $\mathcal{K}_1^f = \{U_1^f \mid U_1 \in \mathcal{K}_1\}$. Clearly \mathcal{K}_1^f is an A_i -class of G -induced π -Hall subgroups.

Let h_1, \dots, h_s be a right transversal of H by $N_H(A_1)$, with $h_1 = 1$. Up to the renumbering we may assume that $A_i = A_1^{h_i}$. Since $AC_G(A) \leq N_G(A_i)$ for every $i = 1, \dots, s$, then elements h_1, \dots, h_s forms a right transversal of G by $N_G(A_1)$. We shall show that the map

$$\mathcal{K}_1 \mapsto \mathcal{K}_1^{h_1} \times \cdots \times \mathcal{K}_1^{h_s}$$

is a bijection between the set of A_1 -classes of G -induced π -Hall subgroups and the set of A -classes of G -induced π -Hall subgroups. Since $\mathcal{K}_1^{h_1} \times \cdots \times \mathcal{K}_1^{h_s}$ is H -invariant, then by Lemma 11 its elements are G -induced π -Hall subgroups. It is also clear that the map is injective. Let $\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_s$ be an A -class of G -induced π -Hall subgroups. To prove the surjectivity it is enough to show that $\mathcal{K}_i = \mathcal{K}_1^{h_i}$ for every $i = 1, \dots, s$. Since G acts transitively on $\{A_1, \dots, A_s\}$, then there exists an element $g \in G$ such that $A_1^g = A_i$. Let $g = act$, where $a \in A$, $c \in C_G(A)$, and $t \in H$. Then $A_1^t = A_i$ and $t \in N_H(A_1)h_i$. As we have proved, $\mathcal{K}_1^t = \mathcal{K}_1^{h_i}$. By Lemma 11 the class \mathcal{K} is H -invariant, therefore $\mathcal{K}_i = \mathcal{K}_1^t = \mathcal{K}_1^{h_i}$ and $\mathcal{K} = \mathcal{K}_1^{h_1} \times \cdots \times \mathcal{K}_1^{h_s}$. \square

2 Proof of Theorem 3

Statement (1) of the theorem follows from [4, Theorem A]. Statement (2) of the theorem follows from [5, Lemma 5.1 and Corollary 5.4] Thus we need to prove statement (3) of the theorem.

Let π be a set of primes such that $2, 3 \in \pi$. Consider all known simple groups separately.

If $S \simeq \text{Alt}_n$, then the statement of the lemma follows from [5, Theorem 4.3].

In sporadic groups all proper π -Hall subgroups with $2, 3 \in \pi$ are given in [6, Table 2] (note that there is an evident missprint in the table, $\{2, 3, 7\}$ -Hall subgroup of J_1 has the structure $2^3 : 7 : 3$). By using [2], it is easy to see that in all cases $k_\pi(S) \leq 2$.

Let S be a finite group of Lie type over a field of characteristic $p \in \pi$ and H be a π -Hall subgroup of S . By [7, Theorem 1.2] one of the following cases 1–4 holds.

Case 1. $H = S$. Clearly $k_\pi(S) = 1$.

Case 2. H is contained in a Borel subgroup of S . In this case, since Borel subgroups are conjugate and solvable, we have that $k_\pi(S) = 1$.

Case 3. $S \simeq D_n^\varepsilon(2^t) \simeq \text{P}\Omega_{2n}^\varepsilon(q)$, $\varepsilon \in \{+, -\}$, H is a parabolic subgroup with a Levi factor isomorphic to $D_{n-1}^\varepsilon(2^t)$. Since all such parabolic subgroups are conjugate, $k_\pi(S) = 1$.

Case 4. $S \simeq \text{PSL}(V)$ and H is an image in $\text{PSL}(V)$ of a parabolic subgroup of $\text{SL}(V)$, stabilizing a series of subspaces $0 = V_0 < V_1 < \dots < V_s = V$. If n_i is equal to $\dim V_i/V_{i-1}$, and n is equal to $\dim V$, then one of the following subspaces holds

(4.1) n is an odd prime, $s = 2$, $\{n_1, n_2\} = \{1, n-1\}$, $k_\pi(S) = 2$;

(4.3) $n = 4$, $s = 2$, $n_1 = n_2 = 2$, $k_\pi(S) = 1$;

(4.4) $n = 5$, $s = 2$, $\{n_1, n_2\} = \{2, 3\}$, $k_\pi(S) = 2$;

(4.5) $n = 5$, $s = 3$, $\{n_1, n_2, n_3\} = \{1, 2, 2\}$, $k_\pi(S) = 3$;

$$(4.6) \ n = 7, s = 2, \{n_1, n_2\} = \{3, 4\}, k_\pi(S) = 2;$$

$$(4.7) \ n = 8, s = 2, n_1 = n_2 = 4, k_\pi(S) = 1;$$

$$(4.8) \ n = 11, s = 2, \{n_1, n_2\} = \{5, 6\}, k_\pi(S) = 2.$$

Assume at last that S is a finite group of Lie type over a field of characteristic p , $p \notin \pi$ and H is a π -Hall subgroup of S .

If $S \simeq A_n^\varepsilon(q)$, where $n \geq 2$, $\varepsilon \in \{+, -\}$, $A_n^+(q) = A_n(q)$ and $A_n^-(q) = {}^2A_n(q)$, then by [5, Lemma 6.1] we have $k_\pi(S) = 1$. Assume that $S \simeq A_1(q)$. If $\pi \cap \pi(S) \neq \{2, 3\}$ and $\pi \cap \pi(S) \neq \{2, 3, 5\}$, then by [5, Lemma 6.1] we have $k_\pi(S) = 1$.

Assume that $\pi \cap \pi(S) = \{2, 3\}$. Then by [5, Lemma 6.1], either H is contained in the normalizer of a maximal torus (and all such subgroups are conjugate), or H is isomorphic to Alt_4 or Sym_4 and H is a homomorphic image of a primitive absolutely irreducible solvable subgroup of $\text{SL}_2(q)$. From [9, § 21, Theorem 6] it follows that all such subgroups H are conjugate in $\text{PGL}_2(q)$, hence in $\text{PSL}_2(q)$ there exist at most 2 classes of such subgroups. Thus in this case $k_\pi(S) \leq 3$.

Assume now that $\pi \cap \pi(S) = \{2, 3, 5\}$. By [5, Lemma 6.1] it follows that in this case either H is contained in the normalizer of a maximal torus (and all such subgroups are conjugate), or H is isomorphic to $\text{PSL}_2(5) = \text{Alt}_5$. More over since $p \notin \pi$, then p does not divide $|\text{SL}_2(5)|$, in particular, $p \neq 2, 3, 5$. By [2] or [8] it follows that $\text{SL}_2(5)$, being the preimage of H in $\text{SL}_2(q)$, there exist 2 exact irreducible complex representation of degree 2, conjugated by an outer automorphism of $\text{SL}_2(5)$, while Alt_5 has no representations of degree 2. By [10, Theorem 15.3 and Corollary 9.7] and the fact that $\text{SL}_2(5) \leq \text{GL}_2(q)$, there exist precisely two nonequivalent representations of degree 2 over the field \mathbb{F}_q of $\text{SL}_2(5)$, conjugated by an outer automorphism. Therefore, all subgroups H , isomorphic to $\text{PSL}_2(5)$, are conjugate in $\text{PGL}_2(q)$, and so in $\text{PSL}_2(q)$ there exists at most 2 classes of such subgroups. Thus, in this case $k_\pi(S) \leq 3$.

Let S be a group of Lie type distinct from $A_n^\varepsilon(q)$, H is its π -Hall subgroup, and $2, 3 \in \pi$, $p \notin \pi$ (in particular, p does not divide $|H|$ and $|2H|$). If S is not isomorphic to one of the groups $\text{P}\Omega_7(q)$, $\text{P}\Omega_8^+(q)$ or $\text{P}\Omega_9(q)$, then by [5, Lemmas 6.2–6.10], H is contained in the normalizer of a maximal torus and all such π -Hall subgroups are conjugate. If S is isomorphic to one of the groups $\text{P}\Omega_7(q)$, $\text{P}\Omega_8^+(q)$ or $\text{P}\Omega_9(q)$, then by [5, Lemmas 6.2 and 6.4] either H is solvable and contained in the normalizer of a maximal torus (in this case all such π -Hall subgroups are conjugate and the order of a Sylow 7-subgroup is greater than 7^2), or $\pi \cap \pi(S) = \{2, 3, 5, 7\}$ and the pair (S, H) lies in the following list: $(\text{P}\Omega_7(q), \Omega_7(2))$, $(\text{P}\Omega_8^+(q), \Omega_8^+(2))$, $(\text{P}\Omega_9(q), 2.\Omega_8(2)^+ : 2)$. Note that if H is nonsolvable, then the order of its Sylow 7-subgroup is not greater than 7^2 , so there does not exist a π -Hall subgroup contained in the normalizer of a maximal torus. By [11, Lemma 1.7.1] the number of conjugacy classes of subgroups of $\text{GO}_7(q)$, isomorphic to $\Omega_7(2)$, and subgroups of $\text{GO}_8^+(q)$, isomorphic to $2.\Omega_8^+(2)$, is not greater than the number of irreducible representation of these subgroups of degrees 7 and 8 respectively. By using ordinary characters tables of $\Omega_7(2)$ and $2.\Omega_8^+(2)$, given in [2] or [8], we obtain that both these groups has precisely one irreducible complex representation of degree 7 or 8 respectively, while $2.\Omega_7(2)$ and $\Omega_8^+(2)$ have no complex representations of degrees 7 and 8 respectively. By [10, Theorem 15.3 and Corollary 9.7], this statement also holds for representations over \mathbb{F}_q . It follows the inequalities $k_\pi(S) \leq 2$ if $S \simeq \text{P}\Omega_7(q)$ and $k_\pi(S) \leq 4$ if $S \simeq \text{P}\Omega_8^+(q)$. In the case $(S, H) = (\text{P}\Omega_9(q), 2.\Omega_8(2)^+ : 2)$ the group H is contained in the centralizer of an involution and by [12, Proposition 11] all such centralizers are conjugate, so in this case $k_\pi(S) \leq 4$ also.

3 Proof of Theorem 1

Assume that the theorem is not true and G is a counter example of minimal order. Then G contains a π -Hall subgroup H and a normal subgroup A such that HA has no the property C_π . Choose from such subgroups A the minimal by inclusion. Let K be a π -Hall subgroup of HA , that is not conjugate to H in HA . The process of eliminating of G we divide into several steps.

Clearly

(1) $HA = KA$.

(2) A is a minimal normal subgroup of G .

Otherwise let M be a nontrivial normal subgroup of G , that is properly contained in A . Let $\overline{G} = G/M$, and for each subgroup B of G by \overline{B} we denote the group BM/M . By Lemma 9 the group \overline{G} has the property C_π , \overline{H} and \overline{K} are its π -Hall subgroups, \overline{A} is a normal subgroup, $\overline{HA} = \overline{KA}$ and $|\overline{G}| < |G|$. By the minimality of the counter example, G the group \overline{HA} has the property C_π . So subgroups \overline{H} and \overline{K} are conjugate by an element of \overline{A} . This means that subgroups HM and KM are conjugate by an element of A . Without lost of generality we may assume that $HM = KM$. In view of the choice of A , the group HM has the property C_π . But this means that H and K are conjugate by an element of $M \leq A$, a contradiction.

(3) $A \notin C_\pi$. In particular, A is not solvable.

Otherwise by Lemma 7 HA would have the property C_π as an extension of a C_π -group by a π -group.

(4) HA is a normal subgroup of G .

Otherwise $N_G(HA)$ is a proper subgroup of G and by Lemma 10 $N_G(HA) \in C_\pi$. Since G is a counter example of minimal order, $HA \in C_\pi$, a contradiction.

By (2) and (3)

(5) A is a direct product of simple non-Abelian groups S_1, \dots, S_m . The group G acts transitively by conjugation on $\Omega = \{S_1, \dots, S_m\}$.

Let $\Delta_1, \dots, \Delta_s$ be orbits of HA on the set Ω , and let $T_j = \langle \Delta_j \rangle$ for each $j = 1, \dots, s$. By (4) and (5)

(6) G acts transitively by conjugation on $\{T_1, \dots, T_s\}$. The subgroup A is a direct product of T_1, \dots, T_s , and each of T_i is normal in HA .

Let $S \in \Omega$ and T is the subgroup, generated by the orbit from $\Delta_1, \dots, \Delta_s$, that contains S . By Lemmas 13 and 15 it follows that

(7) $k_\pi^G(T) = k_\pi^G(S)$.

By (7) and Lemmas 13 and 14 it follows

(8) $k_\pi^G(A) = (k_\pi^G(T))^s = (k_\pi^G(S))^s$.

Since $1 \leq k_\pi^G(S) \leq k_\pi(S)$, from (8) and by Corollary 4 it follows that

(9) $k_\pi^G(A)$ is a π -number.

By Lemma 11

(10) HA leaves invariant each A -class of G -induced π -Hall subgroups.

Since $G \in C_\pi$,

(11) G acts transitively on the set of A -classes of G -induced π -Hall subgroups.

By (10), the subgroup HA is contained in the kernel of this action. Now, by (11)

(12) $k_\pi^G(A)$ is a π' -number.

From (9) and (12) it follows that

$$(13) \ k_{\pi}^G(A) = 1.$$

Now by Lemma 13

$$(14) \ HA \in C_{\pi}, \text{ a contradiction.}$$

Thus the theorem is proved.

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